

Lifshitz Effects on Vector Condensate Induced by a Magnetic Field

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Abstract

By numerical and analytical methods, we study in detail the effects of the Lifshitz dynamical exponent z on the vector condensate induced by an applied magnetic field in the probe limit. Concretely, in the presence of the magnetic field, we obtain the Landau level independent of z , and also find the critical value by coupling a Maxwell complex vector field and SU(2) field into a (3+1)-dimensional Lifshitz black hole, respectively. The research results show that for both two models with the lowest Landau level, the increasing z improves the response of the critical temperature to the applied magnetic field even without the charge density, and the analytical results uphold the numerical results. In addition, we find even in the Lifshitz black hole, the Maxwell complex vector model is still a generalization of the SU(2) Yang-Mills model. Furthermore, we construct the square vortex lattice and discuss the implications of these results.

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I. INTRODUCTION

The gauge/gravity duality [1–3] allows us to deal with a strongly coupled conformal field theory by using its dual weak gravity. Recently, this holographic correspondence has been widely applied to study the high temperature superconductors supposing to involve the strong interaction.

The holographic s-wave superconductor model was first realized via an Einstein Maxwell complex scalar field in a 4-dimensional Schwarzschild anti-de Sitter(AdS) black hole [4]. By the SU(2) gauge field in the AdS black hole, ref. [5] constructed the holographic p-wave superconductor. In the p-wave model, as the temperature is below a critical value, the black hole becomes instable to develop a vector “hair”, which breaks the U(1) symmetry as well as the spatial rotation symmetry spontaneously. The vector condensate is dual to a non-trivial vacuum expected value of vector operator in the boundary field theory, which therefore mimics the p-wave superconductor in the condensed matter system. The holographic d-wave model was built up by using a charged spin two field propagating in the bulk [6]. Near the critical point, the holographic superconductor model was also studied via the analytical Sturm-Liouville (SL) eigenvalue method in ref. [7].

Recently, a new p-wave superconductor model has been constructed by coupling Maxwell complex vector field into the 4-dimensional Schwarzschild AdS black hole in the probe limit [8]. In this model, an applied magnetic field can induce the condensate even without the charge density; Interestingly, the response of this system to the magnetic field is opposite to the one of ordinary superconductors [9, 10], but is quite similar to the case of QCD vacuum phase transition [11, 12]; Moreover, the triangular vortex lattice structure was reproduced in the $x - y$ plane perpendicular to the magnetic field, which was first observed in ref. [13]. Taking into account the back reaction of the matter field, this p-wave model exhibited a rich phase structure [14], which was then observed from the perspective of the holographic entanglement entropy [15]. Whereafter, the holographic p-wave insulator/superconductor phase transition was modeled by introducing such Maxwell complex vector field into the 5-dimensional AdS soliton in the probe limit [16]. It was shown that the Einstein Maxwell complex vector model is a generalization of the SU(2) model with a general mass and gyromagnetic ratio. Very recently, the phase diagrams of this p-wave superconductor model have been further studied in the five dimensional soliton and black hole backgrounds [17]. However, all these holographic models were constructed only in the relativistic spacetimes. Thus we wonder whether the above results still hold in non-relativistic spacetimes, for example, Lifshitz spacetime, which is our motivation in this paper.

As we knew, many condensed matter systems exhibit the anisotropic scaling of spacetime being

characterized by the dynamical critical exponent z as $t \rightarrow b^z t, x^i \rightarrow b x^i$ ($z \neq 1$). This is the so-called Lifshitz fixed points. The authors of [18] proposed the $D = d + 2$ dimensional gravity description dual to this scaling as $ds^2 = L^2 \left(-r^{2z} dt^2 + r^2 \sum_{i=1}^d dx_i^2 + \frac{dr^2}{r^2} \right)$, where $r \in (0, \infty)$ and L the radius of curvature. An alternative way to realize the Lifshitz spacetime is from a massless scalar field coupled to an Abelian gauge field with the action [19]

$$\mathcal{S} = \frac{1}{16\pi G_{d+2}} \int d^{d+2}x \sqrt{-g} \left(R - 2\Lambda - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{4} e^{b\varphi} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right). \quad (1)$$

The Lifshitz spacetime is generalized to a finite temperature system as [20]

$$ds^2 = L^2 \left(-r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \sum_{i=1}^d dx_i^2 \right), \quad (2)$$

where

$$f(r) = 1 - \frac{r_+^{z+d}}{r^{z+d}}, \quad \Lambda = -\frac{(z+d-1)(z+d)}{2L^2}, \quad (3)$$

$$\mathcal{F}_{rt} = \sqrt{2L^2(z-1)(z+d)} r^{z+d-1}, \quad e^{b\varphi} = r^{-2d}, \quad b^2 = \frac{2d}{z-1}. \quad (4)$$

The Hawking temperature of the black hole is given by

$$T = \frac{(z+d)r_+^z}{4\pi}, \quad (5)$$

where r_+ denotes the black hole horizon. To see how the Lifshitz dynamical critical exponent z effects on the superconductor transition, it is helpful to build holographic superconductors in the Lifshitz black hole background. For the related work, see, for example, refs. [21–25]. The scalar condensate was studied in the 4-dimensional Lifshitz black hole with $z = 3/2$ [21] and $z = 2$ [22, 23]. Especially, the author in ref. [23] realized the s-wave and p-wave superconductor models in the 4-dimensional Lifshitz black hole (2). By numerical and analytical methods, we studied the holographic superconductors in 4- and 5-dimensional Lifshitz black hole spacetimes (2) in ref. [24]. It was found that as z increases, the phase transition becomes difficult and the superconductivity becomes weak. However, the critical exponent for the superconductor transition is always one half independent of z and the spacetime dimension. Following ref. [9], in the probe limit, the holographic superconductor with an external magnetic field was studied analytically in ref. [25], and the results showed that the increasing z hinders the scalar condensate while the Lifshitz scaling does not modify the well known Ginzburg-Landau relation for the upper critical magnetic field.

On the basis of the above motivation, in this paper, we will study the Lifshitz effects on the magnetic field induced superconductor transition by coupling the Maxwell complex vector field into

the Lifshitz spacetime (2) on the probe approximation. The results show that in the presence of the magnetic field, the Landau level is obtained, and the vector condensate can be always triggered whether the charge density of the system vanishes or not. Moreover, the increasing z improves the response of the critical temperature T_c to the applied magnetic field with the lowest Landau level. However, the increasing z inhibits the phase transition in the case of non-lowest Landau level. In addition, we also find that even in the Lifshitz spacetime, the SU(2) Yang-Mills model is still the generalization of the Maxwell complex vector model. Last, in the appendix, we will study the condensate of the Maxwell complex vector model without the magnetic field. The results show that below T_c the vector field begins to condensate with a critical exponent 1/2, and the increasing z hinders the phase transition.

This paper is organized as follows. In section II, we study the holographic p-wave superconductor phase transition induced by the applied magnetic field in the Maxwell complex vector model. The holographic p-wave phase transition in the SU(2) Yang-Mills model is studied in section III. The final section is devoted to the conclusions and discussions. In the appendix, we study the condensate of the vector field in the absence of the magnetic field.

II. MAXWELL COMPLEX VECTOR MODEL IN LIFSHITZ SPACETIME

In this section, we study the holographic p-wave superconductor induced by an applied magnetic field in the Lifshitz gravity coupled to the Maxwell complex vector field.

Following ref. [14], we consider the following lagrangian consisting of a Maxwell field and a complex vector field

$$\mathcal{L}_m = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\rho_{\mu\nu}^\dagger\rho^{\mu\nu} - m^2\rho_\mu^\dagger\rho^\mu + iq\gamma\rho_\mu\rho_\nu^\dagger F^{\mu\nu}, \quad (6)$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the strength of the U(1) gauge field A_μ . The tensor $\rho_{\mu\nu}$ is defined by $\rho_{\mu\nu} = D_\mu\rho_\nu - D_\nu\rho_\mu$ with the covariant derivative $D_\mu = \nabla_\mu - iqA_\mu$, and m (q) is the mass (charge) of the vector field ρ_μ . The last term in Eq. (6) represents the interaction between the vector field ρ_μ and the gauge field A_μ , which plays the role of inducing the phase transition. Comparing this lagrangian with that in ref. [26], we ignore the neutral part of the vector meson because it does not contribute to the condensate of charged meson.

In this paper, we neglect the back reaction of the matter sector (6) on the Lifshitz background (2). This is the so-called probe limit, which can be realized by taking $q \rightarrow \infty$ with $q\rho_\mu$ and qA_μ fixed. A variation of the Lagrangian with respect to the vector field ρ_μ provides us with

equation of motion,

$$D^\nu \rho_{\nu\mu} - m^2 \rho_\mu + iq\gamma^\nu F_{\nu\mu} = 0, \quad (7)$$

while the equation of motion for the gauge field can be obtained by varying the Lagrangian with respect to the electromagnetic potential A_μ ,

$$\nabla^\nu F_{\nu\mu} - iq(\rho^\nu \rho_{\nu\mu}^\dagger - \rho^{\nu\dagger} \rho_{\nu\mu}) + iq\gamma^\nu \nabla^\nu (\rho_\nu \rho_\mu^\dagger - \rho_\nu^\dagger \rho_\mu) = 0. \quad (8)$$

To model the magnetic field induced vector condensate, we turn on a magnetic field B perpendicular to the $x - y$ plane as well as a vector field

$$\rho_\nu dx^\nu = (\epsilon \rho_x(r, x) e^{ipy} + \mathcal{O}(\epsilon^3)) dx + (\epsilon \rho_y(r, x) e^{ipy} e^{i\theta} + \mathcal{O}(\epsilon^3)) dy, \quad (9)$$

$$A_\nu dx^\nu = (\phi(r) + \mathcal{O}(\epsilon^2)) dt + (Bx + \mathcal{O}(\epsilon^2)) dy, \quad (10)$$

where ϵ is a small parameter characterizing the deviation from the critical point. $\rho_x(r, x)$, $\rho_y(r, x)$ and $\phi(r)$ are all real functions, and p a constant, while θ is a phase difference between the x and y component of ρ_μ . Substituting the above ansatz into Eq. (8), the equation of $\phi(r)$ at the linear level simplifies to

$$\phi'' + \frac{3-z}{r} \phi' = 0. \quad (11)$$

According to the gauge/gravity dual dictionary, near the boundary $r \rightarrow \infty$, the leading term of the asymptotical expansion for $\phi(r)$ gives the chemical potential μ of the dual theory. To satisfy the norm of A_μ at the horizon, we impose $\phi(r_+) = 0$. Therefore, the gauge field $\phi(r)$ takes the form

$$\phi(r) = \mu \left(1 - \left(\frac{r_+}{r} \right)^{2-z} \right). \quad (12)$$

To seek the solution for ρ_x and ρ_y , we should solve Eq. (7) at the linear order. In order to satisfy the equation of motion with the given ansatz, the phase difference θ can be only chosen as $\theta_+ = \frac{\pi}{2} + 2n\pi$ and $\theta_- = -\frac{\pi}{2} + 2n\pi$ with an arbitrary integer n . Making a variable separation as $\rho_x(r, x) = \varphi_x(r)U(x)$ and $\rho_y(r, x) = \varphi_y(r)V(x)$, at the linear level we have

$$\varphi_x(r) \dot{U}(x) \pm (qBx - p) \varphi_y(r) V(x) = 0, \quad (13)$$

$$\varphi'_x(r) \dot{U}(x) \pm (qBx - p) \varphi'_y(r) V(x) = 0, \quad (14)$$

$$\varphi_x'' + \left(\frac{z+1}{r} + \frac{f'}{f} \right) \varphi_x' - \frac{m^2}{r^2 f} \varphi_x + \frac{q^2 \phi^2}{r^{2(z+1)} f^2} \varphi_x - \frac{(qBx-p)^2}{r^4 f} \varphi_x + \frac{\varphi_x}{r^4 f} \left(\pm(p-qBx) \frac{\dot{V}}{U} \frac{\varphi_y}{\varphi_x} \pm \frac{\gamma qBV}{U} \frac{\varphi_y}{\varphi_x} \right) = 0, \quad (15)$$

$$\varphi_y'' + \left(\frac{z+1}{r} + \frac{f'}{f} \right) \varphi_y' - \frac{m^2}{r^2 f} \varphi_y + \frac{q^2 \phi^2}{r^{2(z+1)} f^2} \varphi_y + \frac{\varphi_y}{r^4 f} \left(\frac{\ddot{V}}{V} \pm (1+\gamma) qB \frac{U}{V} \frac{\varphi_x}{\varphi_y} \pm (qBx-p) \frac{\dot{U}}{V} \frac{\varphi_x}{\varphi_y} \right) = 0, \quad (16)$$

where the prime (dot) denotes the derivative with respect to r (x), while the upper sign and the lower sign correspond to θ_+ and θ_- , respectively. To satisfy Eqs. (13) and (14), we impose the constraints

$$\varphi_y = c\varphi_x, \quad \dot{U} \pm c(qBx-p)V = 0, \quad (17)$$

with a real constant c . Substituting Eq. (17) into Eqs. (15) and (16), we have three equations

$$-\ddot{U} \mp qcB(1+\gamma)V + (qBx-p)^2 U - EU = 0, \quad (18)$$

$$-\ddot{V} \mp \frac{qB(1+\gamma)}{c} U + (qBx-p)^2 V - EV = 0, \quad (19)$$

$$\varphi_x'' + \left(\frac{z+1}{r} + \frac{f'}{f} \right) \varphi_x' - \frac{m^2}{r^2 f} \varphi_x + \frac{q^2 \phi^2}{r^{2(z+1)} f^2} \varphi_x - \frac{E}{r^4 f} \varphi_x = 0, \quad (20)$$

where E is the eigenvalue that can be obtained by solving Eqs. (18) and (19). It is evident that Eqs. (18) and (19) are the same as the ones in ref. [8]. We still take $c^2 = 1$ to make sure that the equations of $U(x)$ and $V(x)$ have exact solutions. Introducing a new function as

$$\psi(x) = U(x) - V(x), \quad (21)$$

and subtracting Eq. (18) from Eq. (19), we can get the equation

$$\psi'' + (E \mp qcB(1+\gamma) - (qBx-p)^2) \psi = 0. \quad (22)$$

Defining a new variable $\xi = \sqrt{|qB|}(x - \frac{p}{qB})$ and constant $\eta = \frac{E \mp qcB(1+\gamma)}{|qB|}$, Eq. (22) simplifies to

$$\psi'' + (\eta - \xi^2) \psi = 0, \quad (23)$$

where the prime denotes the derivative with respect to ξ . The regular and bounded solution to Eq. (23) can be written in terms of the Hermite function

$$\psi(x) = N_n e^{-\frac{1}{2}|qB|(x - \frac{p}{qB})^2} H_n(\sqrt{|qB|}(x - \frac{p}{qB})), \quad (24)$$

as well as the corresponding eigenvalue, i.e., the so-called Landau levels,

$$E_n = (2n + 1)|qB| \pm qcB(1 + \gamma), \quad (25)$$

where N_n and H_n denote normalization constant, the Hermite function, respectively, while n is a non-negative integer. From the eigenvalue (25), we find that the non-minimal coupling γ between the gauge field and the matter field leads to the lowest eigenvalue, which is negative and thus will bring interesting results.

We can read off the effective mass of ρ_μ corresponding to the eigenvalue E_n

$$m_{eff}^2 = m^2 - \frac{q^2\phi^2}{r^{2z}f} + \frac{E_n}{r^2} = m^2 - \frac{q^2\phi^2}{r^{2z}f} + \frac{(2n + 1)|qB| \pm qcB(1 + \gamma)}{r^2}, \quad (26)$$

which depends on the magnetic field B , especially on the non-minimal coupling parameter γ . In what follows, we consider the lowest Landau level ($n = 0$), which reads

$$E_0^L = -|\gamma qB|. \quad (27)$$

Meanwhile, the effective mass of the vector field at the lowest Landau level ($n = 0$) is given by

$$m_{eff}^2 = m^2 - \frac{q^2\phi^2}{r^{2z}f} - \frac{|\gamma qB|}{r^2}. \quad (28)$$

It is clear that the increasing magnetic field decreases the effective mass and thus tends to enhance the superconductor phase transition, while the increasing mass will increase m_{eff}^2 and further makes the phase transition difficult.

Eq. (20) with the effective mass (28) can be written concretely as

$$\varphi_x'' + \left(\frac{z+1}{r} + \frac{f'}{f} \right) \varphi_x' - \frac{m^2}{r^2f} \varphi_x + \frac{q^2\phi^2}{r^{2(z+1)}f^2} \varphi_x + \frac{|\gamma qB|}{r^4f} \varphi_x = 0. \quad (29)$$

Near the boundary $r \rightarrow \infty$, the asymptotical expansion of $\varphi_x(r)$ is of the form

$$\varphi_x(r) = \frac{\varphi_{x-}}{r^{\Delta_-}} + \frac{\varphi_{x+}}{r^{\Delta_+}}, \quad (30)$$

where $\Delta_\pm = \frac{1}{2}(z \pm \sqrt{z^2 + 4m^2})$. According to the AdS/CFT correspondence, φ_{x-} is regarded as the source of the x component of the dual vector operator J_x in the boundary field theory, and φ_{x+} is dual to the expected value of J_x . In order to meet the requirement that the U(1) symmetry should be broken spontaneously, we impose the source-free condition, i.e., $\varphi_{x-} = 0$. From Eq. (30), we can see that the BF bound of the vector field ρ_μ is $m_{BF}^2 = -\frac{z^2}{4}$, at which there exists a logarithmic term in Eq. (30). In this case we take the divergent term as the source to make sure a falloff of φ_x near the boundary.

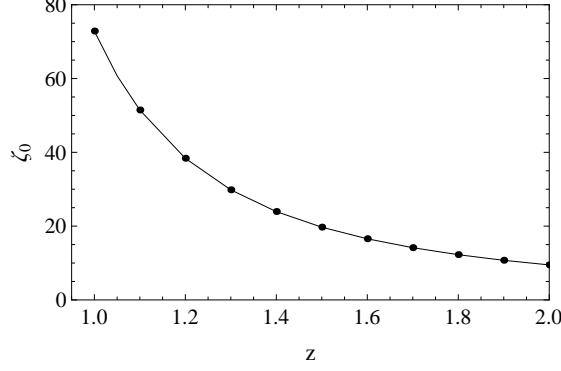


FIG. 1: The critical value ζ_0 versus the Lifshitz dynamical exponent z of the vector field ρ_μ for $\Delta_+ = 3/2$. The black points are obtained by the shooting method to solve Eq. (31).

Now, let us consider the simple case with the vanishing electric field $\phi(r)$. By changing to a dimensionless variable $u = \frac{r_+}{r}$, Eq. (29) can be written as

$$\varphi_x'' - \frac{z-1+3u^{z+2}}{u(1-u^{z+2})}\varphi_x' - \frac{m^2}{u^2(1-u^{z+2})}\varphi_x + \left(\frac{z+2}{4\pi}\right)^{\frac{z}{2}} \frac{\zeta}{1-u^{z+2}}\varphi_x = 0, \quad (31)$$

with $\zeta = |\gamma q B|/T^{2/z}$, where we have used the temperature (5). Near the critical point, we will encounter a marginally stable mode corresponding to the solution of Eq. (31). To solve such the equation, we should impose the boundary conditions, i.e., the regular condition at the horizon, as well as the source-free condition at infinity $\varphi_{x-} = 0$. For a given m^2 and z , only certain special value of ζ can satisfy the equation. Concretely, we first solve this equation by the shooting method.

We plot the critical value ζ_0 as a function of z with fixed $\Delta_+ = \frac{3}{2}$ in Fig. 1. Before analyzing the Lifshitz effect on ζ_0 , it is necessary to determine which side of the phase boundary is superconducting phase. For a given Lifshitz exponent z and the applied magnetic field B , when we decrease the temperature T , the normal phase will become instable to get across the critical point and then enter into the condensed phase, which corresponds to the increasing $\zeta = |\gamma q B|/T^{2/z}$. Therefore the upper right region in the figure stands for the condensed phase while the other region represents the normal phase. From the figure, we find ζ_0 decreases with the increasing z , which indicates that the increasing z enhances the phase transition for the fixed $|\gamma q B|$.

In addition, we show the critical value ζ_0 versus the squared mass m^2 of the vector field with fixed $\Delta_+ = \frac{3}{2}$ in Fig. 2. According to the analysis of $\zeta = |\gamma q B|/T^{2/z}$, we can easily estimate that the upper left region in the figure denotes the condensed phase. From the figure, it is clear that ζ_0 improves with the increase of m^2 . Therefore we can conclude that the increasing m^2 inhibits the superconductor phase transition, which qualitatively agrees with the one in ref. [8].

To uphold above numerical results, we then solve Eq. (31) by the alternatively analytical method,

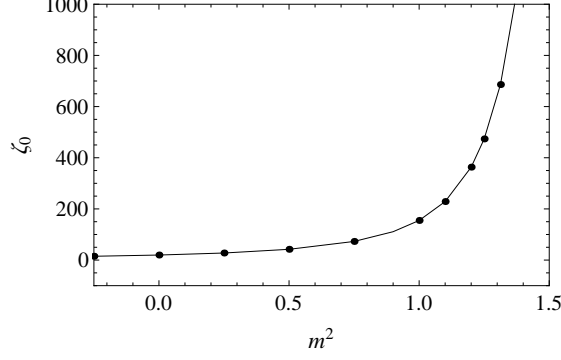


FIG. 2: The critical value ζ_0 with respect to the mass square m^2 of the vector field ρ_μ for $\Delta_+ = 3/2$. The black points are obtained by the shooting method to solve Eq. (31).

i.e., the SL eigenvalue method [7]. By introducing a trial function $\Gamma(u)$ related to $\varphi_x(u)$ as

$$\varphi_x(u) = \langle J_x \rangle \left(\frac{u}{r_+} \right)^{\Delta_+} \Gamma(u), \quad (32)$$

the equation of motion for $\Gamma(u)$ is given by

$$\Gamma''(u) + \frac{(\sqrt{4m^2 + z^2} + z + 3)u^{z+2} - \sqrt{4m^2 + z^2} - 1}{u(u^{z+2} - 1)}\Gamma'(u) - \frac{\frac{1}{2}(z+2)(\sqrt{4m^2 + z^2} + z)u^z - \zeta\left(\frac{z+2}{4\pi}\right)^{\frac{2}{z}} + m^2u^z}{1 - u^{z+2}}\Gamma(u) = 0, \quad (33)$$

with the boundary conditions $\Gamma(0) = 1$ and $\Gamma'(0) = 0$. Such equation can be further written as the SL eigenvalue equation

$$\frac{d}{du} \left(\underbrace{(1 - u^{z+2})u^{\sqrt{4m^2 + z^2} + 1}}_K \Gamma'(u) \right) - \underbrace{\frac{1}{2} \left((z+2)(\sqrt{4m^2 + z^2} + z) + 2m^2 \right) u^{\sqrt{4m^2 + z^2} + z + 1}}_P \Gamma(u) + \underbrace{\zeta \left(\frac{z+2}{4\pi} \right)^{\frac{2}{z}} u^{\sqrt{4m^2 + z^2} + 1}}_Q \Gamma(u) = 0. \quad (34)$$

The minimum eigenvalue of ζ can be obtained by varying the following function

$$\zeta = \frac{\int_0^1 du (K\Gamma'^2 + P\Gamma^2)}{\int_0^1 du Q\Gamma^2}. \quad (35)$$

To estimate the eigenvalue, we take the trial function $\Gamma(u, \alpha) = 1 - \alpha u^2$ with the constant α to be determined. Then we can obtain the minimum value of ζ from Eq. (35) for the given m and z . Here we list the results from the shooting method and the SL method in Tab. I for a clear comparison. It follows that the analytical results are in good agreement with the numerical results.

TABLE I: The critical value of $\zeta = |\gamma q B|/T^{2/z}$ obtained by shooting method and the SL eigenvalue method. For all cases, we fix $\Delta_+ = 3/2$.

	$z = 1$	$z = 6/5$	$z = 7/5$	$z = 8/5$	$z = 9/5$	$z = 2$
shooting	72.842	38.371	23.941	16.569	12.261	9.490
S-L	72.875	38.396	23.962	16.587	12.278	9.506
	$m^2 = -1/4$	$m^2 = 1/4$	$m^2 = 1/2$	$m^2 = 1$	$m^2 = 5/4$	$m^2 = 3/2$
shooting	14.895	27.629	42.096	154.872	473.552	3006.819
S-L	14.912	27.651	42.122	154.921	473.649	3007.164

Especially, when $z = 1$, the results are the same as the ones in ref. [8]. Therefore we can conclude the analytical method is powerful for this Maxwell complex vector model.

Remind that in the literatures, for example, refs. [4, 23], the applied magnetic field was turned off. When the temperature decreases to a critical value, the gauge field A_μ has the non-vanishing mass, which results in the spontaneous breaking of the U(1) symmetry, going with the condensate. Hence, the matter field will not condensate if we turn off the electric field. However, in the presence of the applied magnetic field, even though the electric field vanishes, the instability of the black hole can still be triggered. This interesting result is similar to the QCD vacuum instability, which is induced by the strong magnetic field and develops the condensate of the ρ meson [11, 12]. From the Lagrangian (6), we can see clearly it is the non-minimal coupling term between the vector field ρ_μ and the U(1) gauge field A_μ that leads to the phase transition.

To study systemically the effect of z on the critical value in the system with the applied magnetic field B , next we consider the case with the fixed charge density ρ and the general Landau level E_n . The equation of motion for $\varphi_x(u)$ reads

$$\varphi_x'' - \frac{z-1+3u^{2+z}}{u(1-u^{2+z})}\varphi_x' - \frac{m^2}{u^2(1-u^{2+z})}\varphi_x + \frac{q^2(u^2-u^z)^2\lambda^2}{u^2(1-u^{2+z})^2}\varphi_x - \frac{E_n\lambda}{(1-u^{2+z})\rho}\varphi_x = 0, \quad (36)$$

where $\lambda = \frac{\rho}{r_+^2}$, and E_n is of the form (25). For convenience, we introduce a new function $F(u)$

$$\varphi_x(u) = \left(\frac{u}{r_+}\right)^{\Delta_-} F(u). \quad (37)$$

Then we can get

$$F'' + \left(\frac{2\Delta_-}{u} + \frac{3u^{z+2} + z - 1}{u(u^{z+2} - 1)}\right)F' + \frac{m + \Delta_- (z - \Delta_- + (\Delta_- + 2)u^{z+2})}{u^2(u^{z+2} - 1)}F + \left(\frac{\lambda^2 q^2 (u^2 - u^z)^2}{u^2(u^{z+2} - 1)^2} + \frac{E_n\lambda}{\rho(u^{z+2} - 1)}\right)F = 0. \quad (38)$$

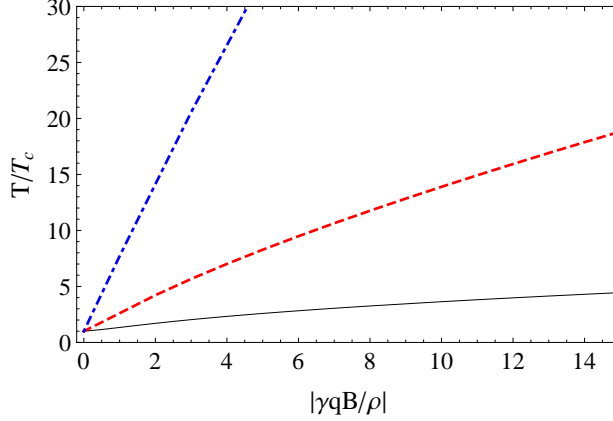


FIG. 3: The critical temperature versus the magnetic field with different z and the fixed $\Delta_+ = 3/2$ for the lowest Landau level ($E_0^L = -|\gamma q B|$). The lines from bottom to top correspond to $z = 1$ (black and solid), $3/2$ (red and dashed), $9/5$ (blue and dotdashed), respectively.

It is easy to see that this equation depends on two dimensionless parameters, i.e., E_n/ρ and λ . Under the regular condition at the horizon ($u = 1$) and the source-free condition near the boundary ($u = 0$), only these two parameters satisfy certain relation, does the equation has the non-trivial solution.

We still consider the case of the lowest Landau level ($E_0^L = -|\gamma q B|$) with fixed $\Delta_+ = 3/2$. The critical temperature T_c as a function of the magnetic field B with different z is plotted in Fig. 3, from which we have the following comments: for the fixed z , when $|\gamma q B/\rho|$ increases, T_c increases, this is obvious from the effective mass that the increasing B decreases m_{eff}^2 and thus raises the critical temperature; For the fixed $|\gamma q B/\rho|$, when z increases ($z = 1, 3/2, 9/5$), the ratio T/T_c increases, which means that the effect of the external magnetic field on T_c becomes more obvious. So far, we can summarize that in the case of the lowest Landau level, the increasing z improves the response of T_c to the applied magnetic field regardless of the charge density. Figure 3 is similar to the one in ref. [27] where the chiral critical temperature improves with the increase of magnetic field.

In addition, we also plot the ratio T/T_c versus $|\gamma q B/\rho|$ for $E_n = |\gamma q B|$ in Fig. 4, to compare with the case for $E_0^L = -|\gamma q B|$. It follows that the increasing magnetic field hinders the phase transition, which is the common property of the ordinary superconductor [9, 10]. Moreover, the fact that the increasing z decreases the critical temperature means that in the case of the non-lowest Landau level, the increasing z hinders the conductor/superconductor phase transition, which is similar to the Lifshitz effect on the Maxwell complex vector model with only the scalar potential

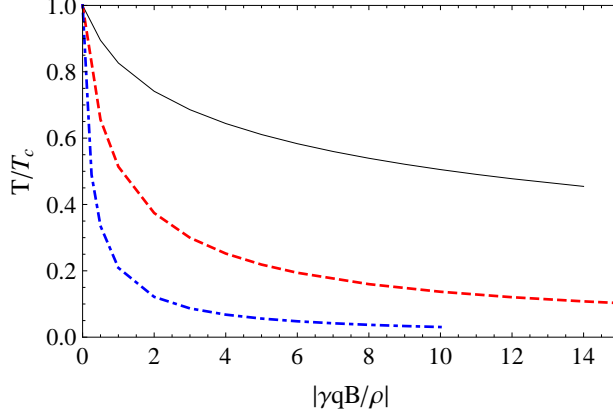


FIG. 4: The critical temperature as a function of the magnetic field with different z and the fixed $\Delta_+ = 3/2$ for the non-lowest Landau level ($E_n = |\gamma q B|$). The lines from top to bottom correspond to $z = 1$ (black and solid), $3/2$ (red and dashed), $9/5$ (blue and dotdashed), respectively.

A_t turned on. See the appendix for the detail calculation of T_c about the complex vector model without the magnetic field.

III. YANG-MILLS MODEL IN LIFSHITZ SPACETIME

The author of ref. [28] discussed the holographic superconductor phase transition induced by the non-Abelian magnetic field in the black hole background. To compare our Maxwell complex vector model with the non-Abelian model, in this section, we study the p-wave superconductor phase transition triggered by the magnetic field in the Lifshitz black hole coupled to a SU(2) field in the probe limit. The SU(2) Yang-Mills Lagrangian is given by [5]

$$\mathcal{L}_m = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}, \quad (39)$$

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \varepsilon^{abc} A_\mu^b A_\nu^c$ is the field strength of the gauge field. The SU(2) group has three generators τ^i which satisfy the commutation relation $[\tau^i, \tau^j] = \varepsilon^{ijk} \tau^k$ ($i, j = 1, 2, 3$). The equation of motion of the gauge field $A = A_\mu^a \tau^a dx^\mu$ reads

$$\nabla_\mu F^{a\mu\nu} + \varepsilon^{abc} A_\mu^b F^{c\mu\nu} = 0. \quad (40)$$

Concretely, we take the ansatz for the gauge field as

$$\begin{aligned} A_\mu^1 dx^\mu &= (\epsilon a_x^1(r, x, y) + \mathcal{O}(\epsilon^3)) dx + (\epsilon a_y^1(r, x, y) + \mathcal{O}(\epsilon^3)) dy, \\ A_\mu^2 dx^\mu &= (\epsilon a_x^2(r, x, y) + \mathcal{O}(\epsilon^3)) dx + (\epsilon a_y^2(r, x, y) + \mathcal{O}(\epsilon^3)) dy, \\ A_\mu^3 dx^\mu &= (\phi(r) + \mathcal{O}(\epsilon^2)) dt + (Bx + \mathcal{O}(\epsilon^2)) dy, \end{aligned} \quad (41)$$

where ϵ is a small parameter characterizing the deviation from the critical point.

Substituting the ansatz (41) into Eq. (40), we can read off the equations of motion

$$\partial_x W_x + (\partial_y - iBx)W_y = 0, \quad (42)$$

$$\partial_x \partial_r W_x + (\partial_y - iBx)\partial_r W_y = 0, \quad (43)$$

$$\begin{aligned} \partial_r^2 W_x + \left(\frac{z+1}{r} + \frac{f'}{f} \right) \partial_r W_x + \frac{1}{r^4 f} \left((\partial_y^2 - 2iBx\partial_y + \frac{\phi^2}{r^{2z-2}f} - B^2 x^2) W_x + \right. \\ \left. (-\partial_x \partial_y + iBx\partial_x - iB) W_y \right) = 0, \end{aligned} \quad (44)$$

$$\begin{aligned} \partial_r^2 W_y + \left(\frac{z+1}{r} + \frac{f'}{f} \right) \partial_r W_y + \frac{1}{r^4 f} \left((\partial_x \partial_y - iBx\partial_x + 2iB) W_x + \right. \\ \left. (\partial_x^2 + \frac{\phi^2}{r^{2z-2}f}) W_y \right) = 0, \end{aligned} \quad (45)$$

where we have defined $W_x = a_x^1 - ia_x^2$ and $W_y = a_y^1 - ia_y^2$. To solve above four equations, we further take a separable form for W_x and W_y

$$W_x(r, x, y) = \tilde{\varphi}_x(r) \tilde{U}(x) e^{ipy}, \quad W_y(r, x, y) = \tilde{\varphi}_y(r) \tilde{V}(x) e^{ipy} e^{i\theta}, \quad (46)$$

which further yields the equations of motion

$$\tilde{\varphi}_x(r) \dot{\tilde{U}}(x) \pm (Bx - p) \tilde{\varphi}_y(r) \tilde{V}(x) = 0, \quad (47)$$

$$\tilde{\varphi}_x'(r) \dot{\tilde{U}}(x) \pm (Bx - p) \tilde{\varphi}_y'(r) \tilde{V}(x) = 0, \quad (48)$$

$$\begin{aligned} \tilde{\varphi}_x'' + \left(\frac{z+1}{r} + \frac{f'}{f} \right) \tilde{\varphi}_x' + \frac{\phi^2 \tilde{\varphi}_x}{r^{2(z+1)} f^2} + \frac{\tilde{\varphi}_x}{r^4 f} \left(- (Bx - p)^2 \pm \right. \\ \left. (p - Bx) \frac{\dot{\tilde{V}}}{\tilde{U}} \frac{\tilde{\varphi}_y}{\tilde{\varphi}_x} \pm \frac{B\tilde{V}}{\tilde{U}} \frac{\tilde{\varphi}_y}{\tilde{\varphi}_x} \right) = 0, \end{aligned} \quad (49)$$

$$\tilde{\varphi}_y'' + \left(\frac{z+1}{r} + \frac{f'}{f} \right) \tilde{\varphi}_y' + \frac{\phi^2 \tilde{\varphi}_y}{r^{2(z+1)} f^2} + \frac{\tilde{\varphi}_y}{r^4 f} \left(\frac{\ddot{\tilde{V}}}{\tilde{V}} \pm 2B \frac{\tilde{U}}{\tilde{V}} \frac{\tilde{\varphi}_x}{\tilde{\varphi}_y} \pm (Bx - p) \frac{\dot{\tilde{U}}}{\tilde{V}} \frac{\tilde{\varphi}_x}{\tilde{\varphi}_y} \right) = 0, \quad (50)$$

where the dot and the prime denote the derivative with respect to x and r , respectively, while the upper sign and the lower sign correspond to the phase difference $\theta_+ = \frac{\pi}{2}$ and $\theta_- = -\frac{\pi}{2}$, respectively. As we all know, for the Maxwell complex vector field model in the standard Schwarzschild AdS black hole and soliton backgrounds [8, 16], in the case of $c^2 = 1$, the SU(2) model can be understood as a generalization of the Maxwell complex vector model with the parameters $m^2 = 0$, $q = 1$, $\gamma = 1$. By comparing Eqs. (47)-(50) with Eqs. (13)-(16), it is easy to see that even in the anisotropy Lifshitz background, such SU(2) model is still the special case of the Maxwell complex vector model with

TABLE II: The critical value of $\zeta = |B|/T^{2/z}$ calculated by shooting method and the SL eigenvalue method.

	$z = 1$	$z = 6/5$	$z = 7/5$	$z = 8/5$	$z = 9/5$	$z = 2$
shooting	39.012	26.846	21.350	18.514	16.963	16.120
S-L	39.031	26.864	21.368	18.534	16.985	16.146

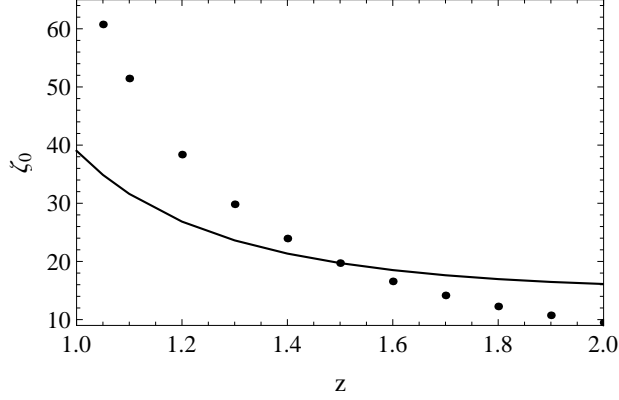


FIG. 5: The critical value ζ_0 with respect to the Lifshitz dynamical exponent z . The black points denote the results from the ρ_μ vector field for comparison.

the parameters chosen as $m^2 = 0$, $q = 1$, $\gamma = 1$, which shows that the convention of $c^2 = 1$ in this paper is still reasonable.

For the lowest Landau level, the effective mass of $\tilde{\varphi}_x(r)$ in the presence of the applied magnetic field and the charge density is given by

$$m_{eff}^2 = -\frac{\phi^2}{r^{2z}f} + \frac{E_n}{r^2} = -\frac{\phi^2}{r^{2z}f} - \frac{|B|}{r^2}. \quad (51)$$

Following the approaches used in the second section, we first consider the simple case with vanishing charge density via the shooting method as well as the SL eigenvalue method. By complicated calculations, we list the results in Tab. II and plot the critical value ζ_0 versus z in Fig. 5. It is easy to see that ζ_0 decreases with the increasing z , i.e., the increasing z enhances the phase transition, which is similar to the case of the Maxwell complex vector model. Moreover, the analytical results agree with the numerical results. Furthermore, the solid line from the non-Abelian magnetic field intersects with the dot line from the Maxwell complex vector field at the value $z = 3/2$, which can be understood as follows: as we all know, in the dual field theory, the power exponent Δ of general falloff for the vector field can be regarded as the “mass” in the field theory. In the Maxwell complex vector field theory, we plot ζ_0 as a function of z with the fixed $\Delta_+ = 3/2$ by adjusting the mass of the vector field. However, in the SU(2) model with the vanishing mass, the power exponent $\Delta = z$

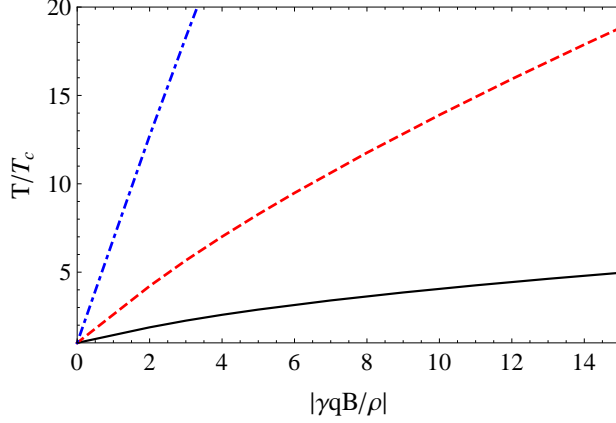


FIG. 6: The critical temperature as a function of the magnetic field with different z for the lowest Landau level ($E_0^L = -|\gamma q B|$). The lines from bottom to top correspond to $z = 1$ (black and solid), $3/2$ (red and dashed), $9/5$ (blue and dotdashed), respectively.

varies with z . When $z = 3/2$, the exponent of the vector operator in the SU(2) field model is the same as that in the Maxwell complex vector field, therefore the critical value from the two models is identical. When $z < 3/2$, the “mass” of the SU(2) field is less than that in the Maxwell complex field, so the critical temperature in the SU(2) field system is larger than that in the latter system.

We also calculate the critical temperature T_c in the SU(2) model with the charge density ρ and the lowest Landau level $E_0^L = -|B|$. The value of T/T_c as a function of the magnetic field is plotted in Fig. 6, from which we can see for the fixed z , T/T_c improves with the increase of B . For the fixed magnetic field, when z increases, T/T_c also increases, which means that the larger z makes the vector condensate easier. These results are very similar to the case of the Maxwell complex vector model. Especially, when $z = 3/2$, the curve from the SU(2) model overlaps with the one from the Maxwell complex vector model, which further proves that the SU(2) field is a generalization of the Maxwell complex vector model. This is obvious from Eq. (30) that $\Delta_+ = 3/2$ and $z = 3/2$ will result in $m^2 = 0$. While the other two cases ($z = 1, 9/5$) have slight difference from the case of the Maxwell complex vector field ($\Delta_+ = 3/2$) due to the different $\Delta_+ = z$ of the vector operator in the SU(2) model.

In addition to the lowest Landau level, we plot the ratio T/T_c versus B with the eigenvalue $E_n = |B|$ in Fig. 7, from which we can see the critical temperature decreases with the increasing magnetic field. Besides, the larger the Lifshitz exponent z is, the smaller the critical temperature is, which implies that the increasing z makes the phase transition more difficult. In particular, the case of $z = 3/2$ in Fig. 7 is still identical with the case of the Maxwell complex vector model with

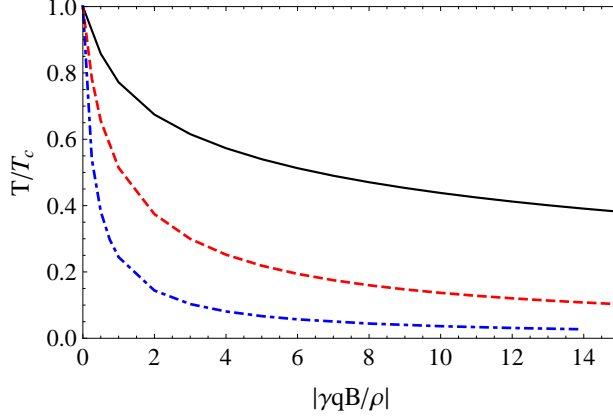


FIG. 7: The critical temperature as a function of the magnetic field with different z for the non-lowest Landau level ($E_n = |\gamma q B|$). The lines from top to bottom correspond to $z = 1$ (black and solid), $3/2$ (red and dashed), $9/5$ (blue and dotdashed), respectively.

$z = 3/2$ in Fig. 4. All these results are similar to the ones of the ordinary superconductor, such as in refs. [8–10, 25].

IV. VORTEX LATTICE SOLUTION

Following ref. [13], we will construct the vortex lattice for the Maxwell complex vector field by superposing the droplet solution in this section. Typically, we only consider the droplet solution with the lowest Landau level ($n = 0$).

Combining Eqs. (21), (24) and (17), we can get the exact solution for $U(x)$ and $V(x)$. The eigen function with the lowest Landau level (27) is

$$U_0^L(x; p) = \frac{N_0}{2} e^{-\frac{|qB|}{2}(x - \frac{p}{qB})^2} = -V_0^L(x; p). \quad (52)$$

Since the eigenvalue does not depend on the constant p , the linear superposition of the solutions $e^{ipy} \varphi_{xn}(r) U_n(x; p)$ and $e^{ipy} \varphi_{yn}(r) V_n(x; p)$ is still the solution of the Maxwell complex vector model at the linear order $\mathcal{O}(\epsilon)$. To obtain the vortex lattice solution from the signal droplet solution (52), we consider the following superposition

$$\begin{aligned} R_l(r, x, y) &= \varphi_{x0}(r) \sum_{-\infty}^{+\infty} C_l e^{ip_l y} U_0^L(x; p_l) - c \varphi_{y0}(r) \sum_{-\infty}^{+\infty} C_l e^{ip_l y} V_0^L(x; p_l) \\ &= \varphi_{x0}(r) \sum_{-\infty}^{+\infty} C_l e^{ip_l y} \psi_0(x; p_l), \end{aligned} \quad (53)$$

with $C_l = e^{-i\pi a_2 l^2 / a_1^2}$ and $p_l = 2\pi \sqrt{|qB|} l / a_1$, where a_1 and a_2 are arbitrary constants, and we have also used convention $c^2 = 1$ and the definition (21). Comparing the vortex lattice solution (53)

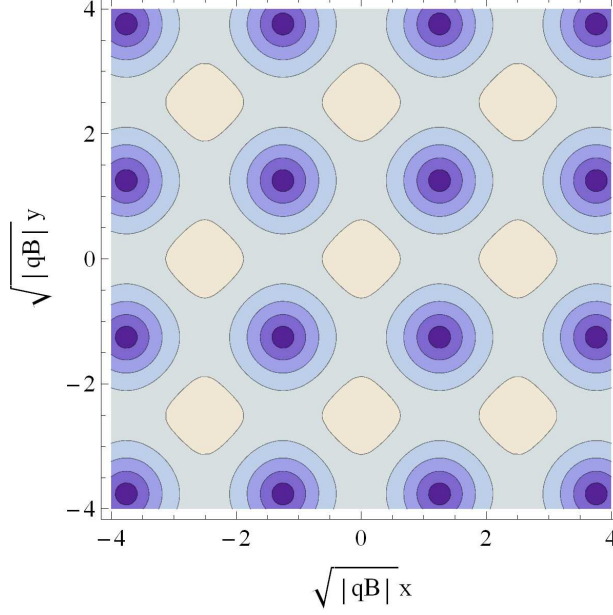


FIG. 8: The contour plot of the square vortex lattice in the $x - y$ plane. Each darker region denotes the core of the vortex, where the lattice solution vanishes.

with the elliptic theta function [13], we can see the vortex lattice solution R_l has two properties. The first property is the pseudoperiodicity of the solution

$$R_l(r, x, y) = R_l(r, x, y + \frac{a_1}{\sqrt{|qB|}}),$$

$$R_l(r, x + \frac{2\pi}{a_1\sqrt{|qB|}}, y + \frac{a_2}{a_1\sqrt{|qB|}}) = e^{\frac{2\pi i}{a_1}(\sqrt{|qB|}y + \frac{a_2}{2a_1})} R_l(r, x, y). \quad (54)$$

The other one is that the cores (or the zeros) of the vortices are located at $\mathbf{x}_{m,n} = (m + \frac{1}{2})\mathbf{b}_1 + (n + \frac{1}{2})\mathbf{b}_2$, where the two vectors $\mathbf{b}_1 = \frac{a_1}{\sqrt{|qB|}}\partial_y$ and $\mathbf{b}_2 = \frac{2\pi}{a_1\sqrt{|qB|}}\partial_x + \frac{a_2}{a_1\sqrt{|qB|}}\partial_y$, while m and n are two integers. Since the expected value of the vector operator J_μ is proportional to subleading order coefficient of the asymptotical expansion of ρ_μ , the combination $J_\pm = \langle J_x \pm iJ_y \rangle$ shows the vortex lattice structure, which is same as that in ref. [8]. Especially, the lowest Landau level corresponds to the case of J_- for $q > 0$ and J_+ for $q < 0$. The SU(2) model is the special case of $q = 1$ for the Maxwell complex vector field, therefore its vortex lattice solution corresponds to J_- .

The triangular lattice with the parameter $a_1 = 2\sqrt{\pi}/3^{1/4}$, $a_2 = 2\pi/\sqrt{3}$ is shown in ref. [8]. For the vector field condensate specializing a spatial direction, the square lattice is also possible. By choosing the parameters $a_1 = \sqrt{2\pi}$ and $a_2 = 1/10000$, the configuration of the condensate $\langle J_- \rangle$ in the $x - y$ plane for the square lattice is plotted in Fig. 8. If we choose other combination of the parameters for a_1 and a_2 , the different structure will be exhibited. However, to determine which case is the true ground state of the system, we should calculate the free energy of the system which

includes the nonlinear effects of the holographic superconductor. By minimizing the free energy, one can obtain the combination of parameters for a_1 and a_2 of the ground state as discussed in ref. [13].

V. CONCLUSIONS AND DISCUSSIONS

So far, in the probe limit, we have studied the properties of the holographic vector condensate induced by an external magnetic field in the 4-dimensional Lifshitz black hole background by numerical and analytical methods. Not only have we discussed the p-wave superconductor phase transition by introducing the Maxwell complex vector field and the $SU(2)$ gauge field in the bulk, respectively, but also emphasized the influence of the dynamical critical exponent z on the critical value. Main conclusions can be summarized as follows.

For the Maxwell complex vector model, it was found that the vector condensate can be induced by the applied magnetic field even without the charge density, and the Landau level is independent of z . Especially, in the case of the lowest Landau level $E_0^L = -|\gamma q B|$, as z increases, the response of the ratio T/T_c to the applied magnetic field becomes more obvious, which means that the increasing z enhances the superconducting phase transition. However, in the case of non-lowest Landau level, for example, $E_n = |\gamma q B|$, the results are opposite to the case of $E_0^L = -|\gamma q B|$, i.e., the increasing z (and the increasing magnetic field B) hinders the phase transition. For the $SU(2)$ Yang-Mills model, we found that even in the Lifshitz spacetime, the Maxwell complex vector field is still a generalization of the $SU(2)$ model with the general mass m , the charge q and the gyromagnetic ratio γ . Due to the diamagnetic and Pauli pair breaking effect of the magnetic field, the results in the case of $E_n = |\gamma q B|$ model the ordinary superconductors, while the result for $E_0^L = -|\gamma q B|$ is quite similar to the case of QCD vacuum instability induced by strong magnetic field to spontaneously develop the ρ -meson condensate [11, 12]. It is worth noting that some studies [29, 30] suggested that the magnetic field can give rise to the superconductor phase transition.

In addition, we also discussed the vector condensate without the applied magnetic field in the appendix. Working in the probe limit, we have found that near T_c , the vector field starts to condensate via a second order phase transition. Moreover, when z increases, the critical temperature decreases, which means that the increasing z hinders the superconducting phase transition.

It should be stressed that we focused on our study in the Lifshitz black hole background on the probe approximation. To see comprehensively the Lifshitz effect on this p-wave superconductor model, it is helpful to extend our present calculations into the soliton background with the Lifshitz

fixed point, which models the p-wave insulator/superconductor phase transition. Furthermore, going beyond the probe limit, a rich phase structure was found for this p-wave model in the absence of the external magnetic field [14, 15, 17]. Hence, in order to understand further the Lifshitz influence on the complete phase diagrams of this vector model, we are to study the back reaction of the Maxwell complex vector field on the Lifshitz black hole in the near future.

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Appendix: Condensate of the complex vector field

Because the charged vector field is dual to a vector operator in the boundary field theory, this Maxwell complex vector field can be regarded as an order parameter to model the p-wave superconductor phase transition. Here we consider the condensate of the complex vector field in the absence of the external magnetic field.

Without loss of generality, we take the ansatz of the matter and gauge field sector as

$$\rho_\nu dx^\nu = \rho_x(r)dx + \rho_y(r)dy, \quad A_\nu dx^\nu = \phi(r)dt. \quad (\text{A.1})$$

Substituting the above ansatz into Eqs. (8) and (7), we can get the following equations of motion

$$\phi'' + \frac{3-z}{r}\phi' - \frac{2q^2}{r^4 f}(\rho_x^2 + \rho_y^2)\phi = 0, \quad (\text{A.2})$$

$$\rho_x'' + \left(\frac{z+1}{r} + \frac{f'}{f}\right)\rho_x' + \left(\frac{q^2}{r^{2z+2}f^2}\phi^2 - \frac{m^2}{r^2 f}\right)\rho_x = 0, \quad (\text{A.3})$$

$$\rho_y'' + \left(\frac{z+1}{r} + \frac{f'}{f}\right)\rho_y' + \left(\frac{q^2}{r^{2z+2}f^2}\phi^2 - \frac{m^2}{r^2 f}\right)\rho_y = 0, \quad (\text{A.4})$$

where the prime denotes the derivative with respect to r . If we choose the ρ_x component to condensate, ρ_y is imposed to be vanishing, which leaves us two coupled differential equations. To solve these equations, we first impose boundary conditions. At the horizon, to make sure the finite form of $g^{\mu\nu}A_\mu A_\nu$, the gauge field should satisfy $\phi(r_+) = 0$, while the vector field requires to be

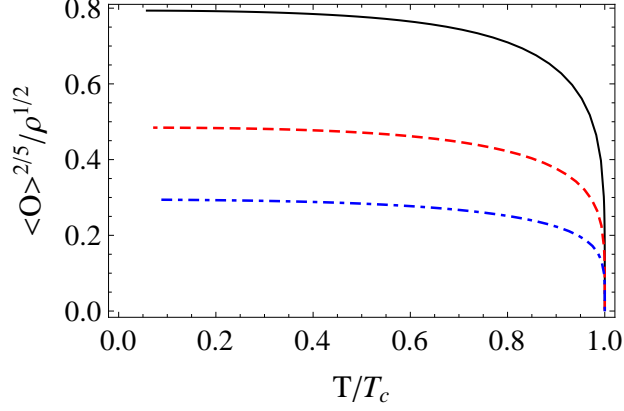


FIG. 9: The condensate versus temperature with different z . The lines from top to bottom correspond to $z = 1$ (black and solid), $z = 3/2$ (red and dashed), and $z = 9/5$ (blue and dotdashed), respectively.

regular at the horizon. Near the boundary $r \rightarrow \infty$, the general falloffs of $\phi(r)$ and ρ_x read

$$\phi(r) = \mu - \frac{\rho}{r^{2-z}} + \cdots, \quad (\text{A.5})$$

$$\rho_x(r) = \frac{\rho_{x-}}{r^{\Delta_-}} + \frac{\rho_{x+}}{r^{\Delta_+}} + \cdots, \quad (\text{A.6})$$

where $\Delta_{\pm} = \frac{1}{2}(z \pm \sqrt{z^2 + 4m^2})$. According to gauge/gravity dual dictionary, μ and ρ correspond to the chemical potential and the charge density in the dual field theory, while ρ_{x-} and ρ_{x+} to the source and the expected value of the boundary operator J_x , respectively. In order to meet the requirement that the symmetry of the system should be spontaneously broken, we turn off the source term, i.e., $\rho_{x-} = 0$.

There is an important symmetry in the system, which is of the form

$$r \rightarrow br, \quad \rho_{x+} \rightarrow b^{\Delta_++1} \rho_{x+}, \quad \rho \rightarrow b^2 \rho, \quad T \rightarrow b^z T, \quad (\text{A.7})$$

with a constant b . By using this symmetry, we can fix the charge density of the system and then work in the canonical ensemble.

The condensate ρ_x as a function of the temperature T is plotted in Fig. 9, from which we find that there is a critical temperature T_c for all cases ($z = 1, 3/2, 9/5$), below which the vector field begins to condensate. The critical temperature for various z is listed as follows

$$z = 1, \quad T_c = 0.102\rho^{\frac{1}{2}}; \quad z = \frac{3}{2}, \quad T_c = 0.043\rho^{\frac{3}{4}}; \quad z = \frac{9}{5}, \quad T_c = 0.014\rho^{\frac{9}{10}}. \quad (\text{A.8})$$

In particular, the result in the case of $z = 1$ restores to the one in ref. [8]. It follows that the larger z hinders the holographic superconductor phase transition, which is similar to the case with the

non-lowest Landau level. In addition, the condensate near T_c for these three cases is fitted as

$$\begin{aligned} z = 1, \quad \frac{\langle J_x \rangle}{\rho^{\frac{5}{4}}} &= 1.13 \sqrt{1 - \frac{T}{T_c}}; & z = \frac{3}{2}, \quad \frac{\langle J_x \rangle}{\rho^{\frac{5}{4}}} &= 0.29 \sqrt{1 - \frac{T}{T_c}}; \\ z = \frac{9}{5}, \quad \frac{\langle J_x \rangle}{\rho^{\frac{5}{4}}} &= 0.08 \sqrt{1 - \frac{T}{T_c}}. \end{aligned} \tag{A.9}$$

Evidently, the critical exponent $\frac{1}{2}$ is universal for all cases, which indicates that near the critical temperature, the system undergoes a second order transition. Besides, as z increases, the efficient of the condensate decreases, which agrees with the fact that T_c decreases with z .

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